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A deterministic sandpile automaton revisited

S. Lübeck[1,a], N. Rajewsky[2], and D.E. Wolf[1]

[1] Theoretische Physik, Gerhard-Mercator-Universität Duisburg, 47048 Duisburg, Germany
[2] Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903, USA

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Abstract. The Bak-Tang-Wiesenfeld (BTW) sandpile model is a cellular automaton which has been intensively studied during the last years as a paradigm for self-organized criticality. In this paper, we reconsider a deterministic version of the BTW model introduced by Wiesenfeld, Theiler and McNamara, where sand grains are added always to one fixed site on the square lattice. Using the *Abelian sandpile* formalism we discuss the static properties of the system. We present numerical evidence that the deterministic model is only in the BTW universality class if the initial conditions and the geometric form of the boundaries do not respect the full symmetry of the square lattice.

PACS. 64.60.Ht Dynamic critical phenomena – 05.65.+b Self-organized systems – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion

1 Introduction

In equilibrium systems with short-ranged interactions and no broken continuous symmetry, correlation functions usually decay exponentially with the distance. Exceptions from this behavior occur only in special cases, for example at critical points in a phase diagram. In this case the corresponding correlation functions decay algebraically if the relevant system parameter is adjusted to special (critical) values (*e.g.* critical temperature, critical pressure, etc.).

The concept of self-organized criticality (SOC) which was introduced by Bak, Tang and Wiesenfeld in 1987 [1,2] attempts to explain the fact that in nature critical behavior is often observed, although nature cannot "fine-tune parameters" (see [3,4] for an introduction and overview). The term "critical behavior" corresponds here to a power-law behavior of the probability distributions of certain physical quantities which characterize the system in both space and time [1,2]. Typical examples for such quantities are the size and life times of catastrophic events. The main idea is then that the critical state is an attractor of the dynamics.

One of the paradigmatic systems which exhibit SOC is the Bak-Tang-Wiesenfeld sandpile model (BTW model) which was intensively investigated in the past. In the following, we restrict our discussion to the BTW model on the two-dimensional square lattice. Concerning static properties, analytical results exist which are mainly due to Dhar, who developed a formalism for *Abelian sandpile models* [6] which allows to calculate exactly the height probabilities, height correlations, number of steady state configurations, etc. [6–9]. However, much less is known rigorously about the dynamical features, and estimates for the exponents of the probability distributions of avalanche quantities are only known from computer simulations (see for instance [10–15]); further guesses for these exponents have been made *via* renormalization group approaches (*e.g.* [16,17]).

In this paper we consider a deterministic version of the BTW model which was introduced by Wiesenfeld *et al.* [18]. In the original BTW model, sand grains are added at randomly chosen sites of the lattice. In the deterministic BTW model (DBTW) the seeding of sand is confined to one special site of the lattice. Computer simulations revealed [18] that the DBTW model still displays criticality. Therefore, randomness in the location of the perturbations is not a necessary ingredient for SOC [18]. Furthermore, the authors concluded from their numerical analysis that the different versions of the BTW model could display different scaling behavior.

However, this conclusion was obtained from an investigation of the DBTW model for a small system size. As finite-size effects have been shown to affect the scaling behavior of the BTW model strongly [11,12], we reinvestigate the case here and present a systematic finite-size analysis. We also present some new exact results for the DBTW model.

The paper is organized as follows. In Section 2, we define the model and, using essentially the formalism developed by Dhar [6], study the static properties of the DBTW. In Section 3, we present our results from computer simulations and then discuss the observed scaling behavior in the context of the universality hypothesis of Ben-Hur and Biham [19]. A summary closes the paper.

[a] e-mail: `sven@thp.uni-duisburg.de`

2 Definition of the model and static properties

The BTW model on a two-dimensional square lattice of size $L \times L$ is defined as follows. To each site i ($i = 1, 2, \ldots, L^2$) an integer variable z_i (the height) is assigned. Starting with an empty lattice ($z_i = 0$ for all i) the addition of a grain of sand means to choose a site i at random and to increase z_i by 1. If z_i is equal to a fixed threshold value z_{c}, site i topples and distributes one grain of sand to each nearest neighbor, which can in turn trigger more toppling events. Thus, an avalanche of relaxation events may take place. For the sake of simplicity (and without loss of generality) we set $z_{\mathrm{c}} = 4$ throughout this work. The toppling rules can be formulated in terms of a $N \times N$ toppling matrix Δ [6], where $N = L^2$. If site i topples, one has

$$z_j \longrightarrow z_j - \Delta_{ij} \tag{1}$$

for all $j = 1, 2, ..., N$, where the toppling matrix satisfies the conditions $\Delta_{ii} = 4$, $\Delta_{ij} = -1$ if i and j are nearest neighbors, and $\Delta_{ij} = 0$ otherwise. Sand falling over the rim of the system is discarded.

In the following we briefly describe the *Abelian sandpile* formalism which was introduced by Dhar [6] and recall the major results. Dhar showed that the dynamics of the BTW model is well-defined in the sense that the resulting stable configuration $C = \{z_i\}$ is always the same, regardless of the order, in which critical sites are updated during an avalanche. One can define an operator a_i by its action onto a stable configuration C: $a_i C$ is the stable configuration which results from adding a particle at site i and relaxing the resulting configuration. Dhar showed that

$$[a_i, a_j] = 0, \tag{2}$$

for all i, j. Therefore, the BTW model is called an *Abelian sandpile model* [6].

One is interested in the stationary state of this cellular automaton, *i.e.*, one iterates the dynamical rules until all expectation values become time independent. Since the dynamics can be described as a Markovian process a stable configuration can either be transient or recurrent. A recurrent configuration C can be defined by demanding that for every possible seeding site i a natural number $m_i(C)$ exists such that $a_i^{m_i(C)} C = C$ holds. Thus, for a recurrent configuration C and for a natural number l one gets [6]

$$a_i^l C = a_i^l a_i^{m_i(C)} C = a_i^{m_i(C)} a_i^l C, \tag{3}$$

which shows that $a_i^l C$ is a recurrent configuration, too. Therefore, the set of recurrent configurations is closed under the action of the operators a_i [6]. Since it is possible to define unique, inverse operators a_i^{-1}, it follows that each recurrent configuration has the same probability to appear in the stationary state. One can further prove that the number of recurrent configurations is equal to $\det \Delta$ [6].

A two point correlation function G_{ij} can be defined in the following way: let G_{ij} be the expectation value for the number of topplings in j, which are caused by adding a particle at site i. In the stationary state, the average number of particles which enter site j must be equal to the average of sand grains leaving site j:

$$G_{ij} \Delta_{jj} = \sum_{k \neq j} G_{ik} (-\Delta_{kj}) + \delta_{ij}, \tag{4}$$

which implies $G = \Delta^{-1}$ [6]. The analytic expression for G is well-known:

$$G_{ij} = {\Delta^{-1}}_{ij} = \frac{1}{(L+1)^2} \sum_{a,b=1}^{L} \frac{\sin x_i \tilde{a} \sin x_j \tilde{a} \sin y_i \tilde{b} \sin y_j \tilde{b}}{\sin^2 \frac{\tilde{a}}{2} + \sin^2 \frac{\tilde{b}}{2}}, \tag{5}$$

with $\tilde{a} = a\pi/(L+1)$, $\tilde{b} = b\pi/(L+1)$ and where the sites i and j have the coordinates (x_i, y_i) and (x_j, y_j), respectively. Let s denote the size of an avalanche, *i.e.*, the total number of topplings during that avalanche. Using equation (5) it can be shown that the average number of topplings $\langle s \rangle$ is given by

$$\langle s \rangle = \frac{1}{L^2} \sum_{i,j} G_{ij} = \frac{1}{L^2 (L+1)^2} \sum_{\substack{a,b \\ \mathrm{odd}}}^{L} \frac{\cot^2 \frac{\tilde{a}}{2} \, \cot^2 \frac{\tilde{b}}{2}}{\sin^2 \frac{\tilde{a}}{2} + \sin^2 \frac{\tilde{b}}{2}} \tag{6}$$

which scales for large L as [6]

$$\langle s \rangle \sim L^2. \tag{7}$$

Let us now turn to the deterministic BTW model (DBTW) as introduced by Wiesenfeld, Theiler, and McNamara [18]. Here, sand is always added at a fixed input site i_0. Thus, the dynamics is fully deterministic and it is clear that in the configuration space (recurrent configurations) a DBTW model will settle down in an orbit with some period T, which in general will depend on i_0. For example, the period of a DBTW model with a seeding site at the corner of the lattice is larger than the period of the center-seeded DBTW model, as it is intuitively clear and also known from computer simulations. An analytical approach [20] was used to evaluate T for the center-seeded DBTW model up to a system of $N = 361$ sites, where the authors found a period of length $\approx 10^{17}$ and extrapolated their results to reproduce the numerical estimation of [18] $T \sim \exp(0.11\, N)$.

In [18], several interesting features of the DBTW model could be derived by using the Abelian sandpile formalism:

(i) the orbits have the same period (for a fixed input site i_0), regardless of the initial conditions;
(ii) the minimal stable configuration $C^* = \{z_i = z_{\mathrm{c}} - 1,$ for all $i\}$ is always on an orbit of the DBTW model.

Note that (ii) does not mean that a DBTW model will always reach C^* at some point. For example, for the initial condition $z_i = 1$ for all i a system of 49 sites has an orbit without C^*. Only for different initial conditions or a different system size (*e.g.* 25 sites with the same initial condition) C^* is on the orbit.

Let us now define $N_{i_0 j}$ as the total number of topplings at site j within the period T of a DBTW model with input site i_0. The time average of $N_{i_0 j}$ is simply $N_{i_0 j}/T$. The total flow of particles during T into j has to equal the flow out of j and it follows $N_{i_0 j}/T = \Delta^{-1}_{i_0 j}$ or

$$\Delta^{-1}_{i_0 j} = G_{i_0 j} = N_{i_0 j}/T. \tag{8}$$

This means that every "row" i of the BTW correlation function G_{ij}, which stands for the ensemble-averaged number of topplings in j when seeding in i, represents the time average of topplings in j of a DBTW model with a fixed input site i_0. The average avalanche size $\langle s \rangle$ for the BTW model can therefore be thought of as $\langle s_{i_0} \rangle$ of the DBTW models averaged over all orbits and all possible seeding sites $i_0 = 1, 2, ..., N$.

Similar to the BTW model it is possible to calculate the average number of topplings $\langle s \rangle$ for the center-seeded DBTW model on a square lattice with length L (L odd). Using equation (5) we get with $x_{i_0} = y_{i_0} = (L+1)/2$

$$\langle s \rangle = \sum_j G_{i_0 j} = \frac{1}{(L+1)^2} \sum_{a,b=0}^{\frac{L-1}{2}} (-1)^{a+b} \frac{\cot \frac{\tilde{A}}{2} \, \cot \frac{\tilde{B}}{2}}{\sin^2 \frac{\tilde{A}}{2} + \sin^2 \frac{\tilde{B}}{2}} \tag{9}$$

with $\tilde{A} = (2a+1)\pi/(L+1)$ and $\tilde{B} = (2a+1)\pi/(L+1)$, respectively. It is straightforward (though tedious) to evaluate this expression for large L by using standard methods. One obtains $\langle s \rangle \sim L^2$, analogous to the BTW model. It is also easy to get an expression for the time-averaged number of topplings $N_{i_0 i_0}/T$ of the input site i_0:

$$\frac{N_{i_0 i_0}}{T} = G_{i_0 i_0} = \frac{1}{(L+1)^2} \sum_{a,b=0}^{\frac{L-1}{2}} \frac{1}{\sin^2 \frac{\tilde{A}}{2} + \sin^2 \frac{\tilde{B}}{2}} \tag{10}$$

which can be shown, again using elementary methods, to scale as $\frac{N_{i_0 i_0}}{T} \sim \ln L$ for large L.

3 Dynamics

3.1 Characterization of the avalanches

For a graphical representation of the avalanches, it is convenient to denote how many topplings n at each site have occurred during the avalanche [10]. Sites with the same number of topplings (during one avalanche) form "shells". It is easy to check that for each site i inside such a shell, for which all four neighbors are also part of the shell, the height z_i before and after the avalanche remains the same. Figure 1 shows some examples which have been obtained in the stationary state of the central seeded DBTW. The shells seem to form compact sets with n monotonically decreasing from the central site towards the boundaries

```
    A A A                 A A   A A
  A B B B B B A         A A A   A A A
  B C C C C C B       A A A A   A A A A
A B C D D D C B A     A A A A A A A A A
A B C D E D C B A           A B A
A B C D D D C B A     A A A A A A A A A
  B C C C C C B       A A A A   A A A A
  A B B B B B A         A A A   A A A
    A A A                 A A   A A

                A       A
                A A   A A
              A A B A B A A
            A A A B B B A A A
          A B B C C C C C B B A
        A B C C D D D D D C C B A
      A B C D D E E E E E D D C B A
    A A B C D D E F F F E D D C B A A
  A A A A C D E E F G G G F E E D C A A A A
  A B B C D E F G H H H G F E D C B B A
    A B C D E F G H J H G F E D C B A
  A B B C D E F G H H H G F E D C B B A
  A A A A C D E E F G G G F E E D C A A A A
    A A B C D D E F F F E D D C B A A
      A B C D D E E E E E D D C B A
        A B C C D D D D D C C B A
          A B B C C C C C B B A
            A A A B B B A A A
              A A B A B A A
                A A   A A
                A       A
```

Fig. 1. Three different avalanches for the center-seeded DBTW model. Sites which toppled once are marked as A, twice toppled sites as B, etc.

along the symmetry axis. Also, it seems that at the boundary of each $n =$ const shell, n will change only by 1 (especially at the boundary of the avalanche one finds $n = 1$ most often). However, the latter statement is not always correct, there are rare exceptions. For instance, the upper left cluster in Figure 1 has a boundary site which has toppled twice and the lower cluster has sites with $n = 1$ adjacent to a site with $n = 3$. We could only prove two properties of the avalanches. First, it has been shown for the BTW model, and the proof applies for the DBTW model also, that avalanches are always compact [10,21]. Second, one can show that n decreases monotonically from the center towards the boundaries along the symmetry axis by decomposing an avalanche into a series of waves of topplings [22,23]. First one topples the center site and relaxes all other sites which become unstable. This defines the first wave of topplings. After this, one allows the center site to relax again (if possible) which generates a second wave of topplings and so forth. It has been shown [22] that each avalanche produced by such a wave of topplings is compact. Furthermore, each site in a wave can topple only once. Thus, by superposition of the compact waves of topplings, n can only monotonically decrease from the center towards the boundaries along the symmetry axis.

3.2 Scaling behavior of the avalanches

In driven systems the boundary conditions can influence the stationary state [24]. We show below that the scaling properties of the BTW model and its deterministic version are only the same if the latter has boundary conditions or an initial configuration which do not respect the square symmetry.

We denote by s the total number of topplings which occurred during the lifetime t (in units of lattice sweeps) of an avalanche. The area a of such an avalanche is the number of distinct toppled sites. The outflow o is the total number of sand grains which leave the system during an avalanche. The linear size of an avalanche is measured *via* the radius of gyration of an avalanche cluster. In the critical steady state the corresponding probability distributions should obey a power-law behavior

$$P_x(x) \;\sim\; x^{-\tau_x}, \tag{11}$$

characterized by the exponent τ_x with $x \in \{s, a, t, o, r\}$. As usual, we measure the avalanche distributions by counting the numbers of avalanches corresponding to a given area, duration, etc. and integrate these numbers over bins of increasing length (see for instance [25]). In our simulations successive bin lengths increase by a factor $b = 1.2$. In the case of the BTW model it is known that the probability distributions display logarithmic corrections

$$P_x(x) \;\sim\; x^{-\tau_x}\, x^{\mathrm{const}/\ln L}, \tag{12}$$

which are caused by finite-size effects [11,12]. A numerical determination of the avalanche exponents requires therefore a careful analysis of finite-size effects. Using the functional form of the finite-size corrections it is possible to determine the exponents τ_x directly, *i.e.* without any extrapolation to the infinite system, and the best known values are $\tau_s = 1.293 \pm 0.009$, $\tau_a = 1.33 \pm 0.011$, $\tau_t = 1.480 \pm 0.011$, and $\tau_r = 1.665 \pm 0.013$ [12].

We performed simulations of the center-seeded DBTW model for various system sizes $L \leq 2049$ and averaged all measurements over at least 2×10^6 avalanches. Starting from an empty lattice, we added particles at the lattice center and applied the "burning algorithm" in order to check if the system has reached the steady state [6]. Thus the initial conditions (as well as the boundary conditions) respect the square symmetry. Figure 2 shows the probability distribution $P_t(t)$ of the avalanche duration. Surprisingly, it turns out that the above mentioned logarithmically averaging method is not suitable in our case, because avalanches of odd or even duration display a different scaling behavior. The two branches of $P_t(t)$ in Figure 2 (τ_t^{odd} and τ_t^{even}) clearly have different slopes for the system size considered. The probability distribution

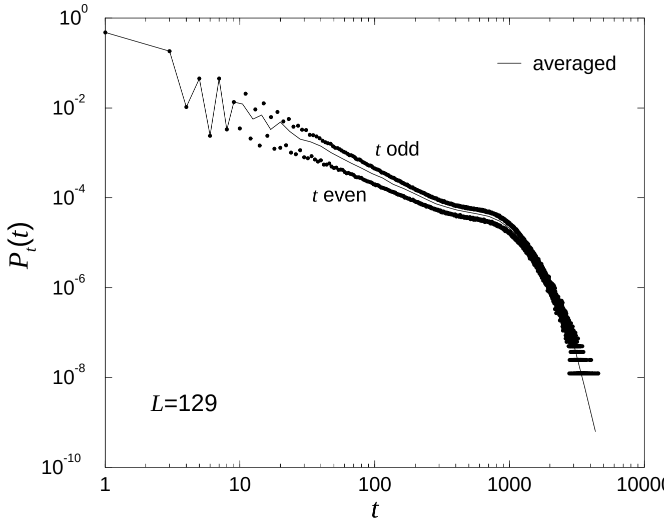

Fig. 2. The probability distribution of the avalanche duration $P_t(t)$. Avalanches of an even or odd duration display a different scaling behavior. Thus, the usual logarithmic averaging of the distribution leads to useless results (solid line).

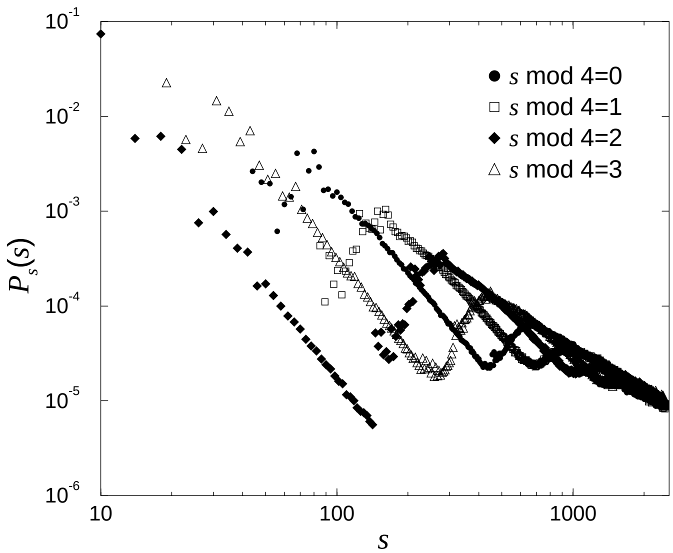

Fig. 3. The probability distribution of the avalanche duration $P_s(s)$. The distribution decomposes into four different branches corresponding to the four possible values of $s \bmod 4$.

of the avalanche size $P_s(s)$ exhibits an even more complicated fine structure of four distinct branches corresponding to the four possible values of $s \bmod 4$ (see Fig. 3). We have no explanation for this behavior.

Let us briefly remark that a similar symmetry effect can be found in the outflow probability distribution $P_o(o)$. One gets different curves for all o which are divisible by 8 and those which are not (see Fig. 4). To explain this, we note that the outflow o is divisible by 8 exactly when the sites in the middle of the boundary edges do *not* topple. This is extremely unlikely for large avalanches. On the other hand, for small avalanches, the constraint to topple sites in the middle of the boundary edges considerably reduces the number of possible avalanches.

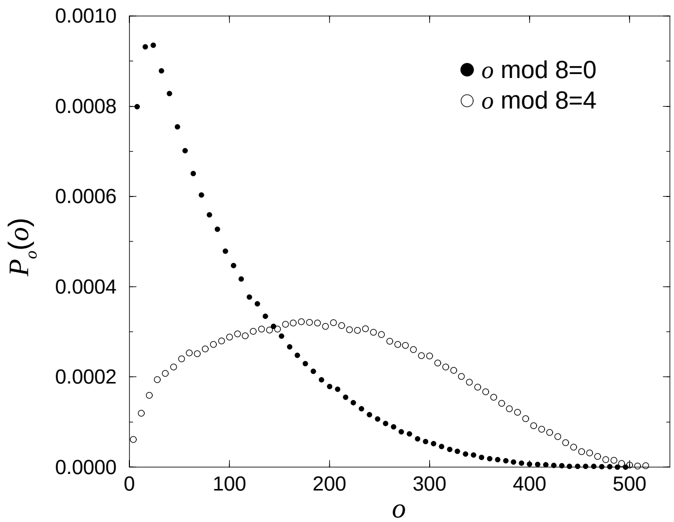

Fig. 4. The probability distribution of the sand outflow $P_o(o)$ for $L = 129$. The distribution decomposes into two different branches corresponding to the two possible values of $o \bmod 8$.

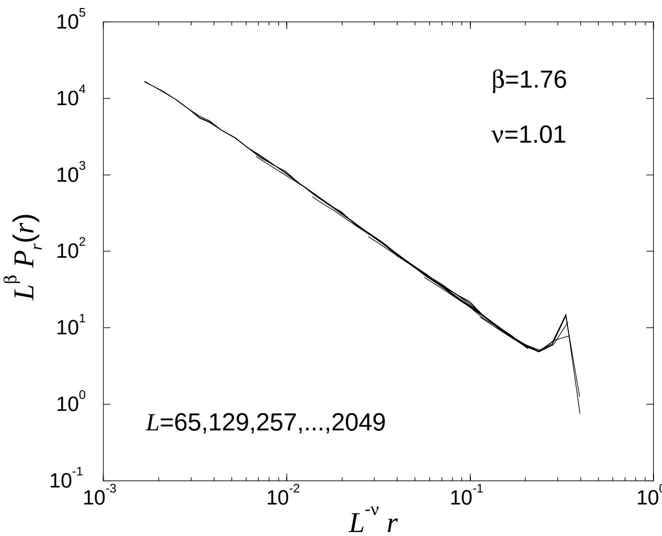

Fig. 6. The finite-size scaling analysis of the avalanche distribution $P_r(r)$. Since the radius scales with the system size the exponent ν_r should equal one.

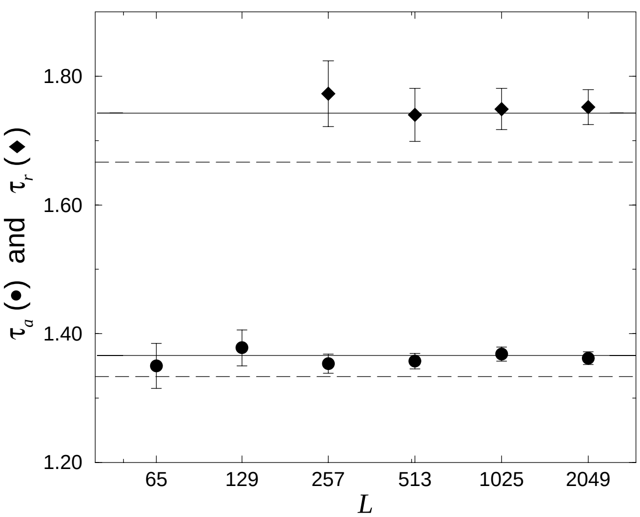

Fig. 5. System size dependence of the avalanche exponents τ_a and τ_r. The solid lines corresponds to the values obtained from a finite-size scaling analysis and the dashed lines corresponds to the values of the BTW model obtained in [12].

Since the exponents of the size and duration distribution, τ_s and τ_t, are not well-defined here we renounce further investigations of the size and duration distribution in this section and focus our attention on the probability distributions of the avalanche area and radius which behave as usual (see below). We measured the probability distributions $P_a(a)$ and $P_r(r)$ for various system sizes and obtained the corresponding exponents from a power-law fit of the straight portion of the curves. The values of both exponents are plotted in Figure 5. In contrast to the BTW model (Eq. (12)) the avalanche exponents of the center-seeded DBTW model display no significant system size dependence. This allows us to apply the finite-size scaling analysis [26]

$$P_x(x, L) = L^{-\beta_x} g_x(x L^{-\nu_x}), \tag{13}$$

where the scaling exponents β_x and ν_x are connected with the avalanche exponent τ_x *via* the scaling equation $\beta_x = \tau_x \nu_x$ [26]. This finite-size scaling ansatz works for the area and radius distribution and the corresponding data collapse for $P_r(r)$ is plotted in Figure 6. The obtained values for the avalanche exponents agree with the results of the regression analysis (see Fig. 5) and we get $\tau_a = 1.368 \pm 0.011$ and $\tau_r = 1.752 \pm 0.027$.

As already mentioned in Section 3.1, it is possible to show that the avalanches are compact. Thus, the area scales with the radius as

$$a \sim r^2. \tag{14}$$

Then, the transformation law of probability distributions $P_a(a)\mathrm{d}a = P_r(r)\mathrm{d}r$ leads to the scaling relation

$$2 = \frac{\tau_r - 1}{\tau_a - 1} \cdot \tag{15}$$

This scaling relation is fulfilled within the error-bars, which confirms the accuracy of the determination of the avalanche exponents τ_a and τ_r. Finally we mention that our obtained exponents are consistent with the values $\tau_a = 11/8$ and $\tau_t = 7/4$ and that these values obey the above scaling relation exactly.

4 Discussion

Since both avalanche exponents differ significantly from those of the BTW model we conclude that the center-seeded DBTW model does not belong to the BTW universality class. It is worth to examine the different universal behavior in detail since the universality hypothesis of

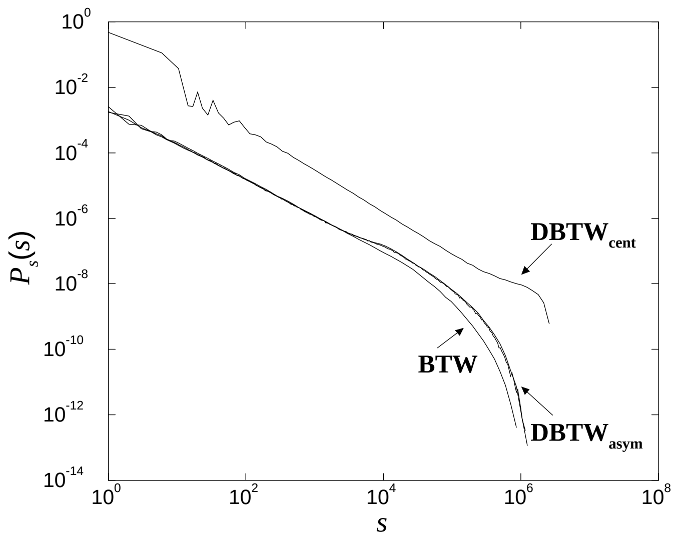

Fig. 7. The probability distribution $P_s(s)$ of the symmetric center-seeded DBTW model, two asymmetric DBTW models (in one case the square symmetry is broken by the boundaries and in the other case by asymmetric initial conditions), and the BTW model for $L = 257$. In the latter cases the curves are shifted in the downward direction.

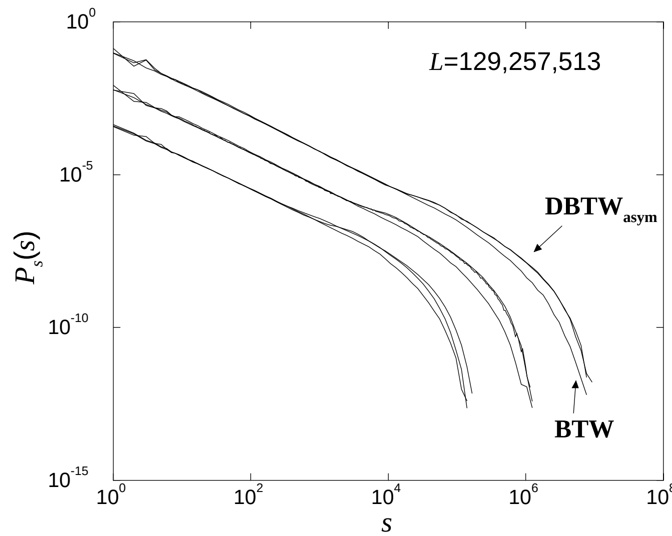

Fig. 8. The probability distribution $P_s(s)$ of the two asymmetric DBTW models and the BTW model for various system sizes. For $L < 513$ the curves are shifted in the downward direction.

Ben-Hur and Biham states that the only parameter which determines the scaling behavior (exponents) of a sandpile model is the so-called relaxation vector which describes how the sand grains of a critical site are distributed to the next neighbors [19]. Applying this concept of classification one can identify three universality classes where the distribution is *nondirected*, *nondirected on average*, and *directed*. For instance, the BTW and the related Zhang model [27] belong to the universality class of nondirected models, whereas the Manna model [25] is nondirected on average and therefore belongs to a different class. Several numerical investigations confirm these classification ansatz (see for instance [12,19,28] and for recent investigations [29,30]).

According to this classification concept the center-seeded DBTW model and the BTW model should belong to the same universality class because both models are characterized by the same relaxation vector. In contrast to the BTW model the center-seeded DBTW model is deterministic and both the avalanches and the height configurations display the square symmetry. Moving the input site i_0 from the lattice center brakes the square symmetry but the dynamics of the system is still deterministic. The same effect is obtained if we use center-seeding but start with an asymmetric initial condition. We call these models the asymmetric DBTW models and our analysis revealed that they display the same scaling behavior as the usual BTW model. We plot the probability distribution $P_s(s)$ for the considered models in Figure 7. Except of deviations at the cut-off, the probability distributions of the asymmetric DBTW models agree with the corresponding curve of the BTW model and differ clearly from the distribution of the symmetric DBTW model. Figure 8 shows the probability distribution for various system sizes.

Again, apart from the cut-off behavior the curves of the BTW model and the asymmetric DBTW models are identical. This implies that the avalanche exponents of the asymmetric DBTW models display the same logarithmic corrections as the BTW model (Eq. (12)).

We conclude from our investigations that the asymmetric DBTW and the BTW model belong to the same universality class, whereas the center-seeded DBTW model with symmetric initial height configuration does not. Since the asymmetric DBTW model is still deterministic but lacks the square symmetry the different universal behavior of the center-seeded DBTW model is not caused by the deterministic dynamics but by the square symmetry of the system (in agreement with [18]).

We conclude from our results that properties of the steady state such as symmetries or translational invariance can affect the universality class of sandpile models. This is confirmed by recently performed simulations [31] of a directed version of the Zhang model which exhibits a different scaling behavior than the exactly solved directed BTW model [32]. According to the classification of Ben-Hur and Biham the directed Zhang model and the directed BTW model should belong to the same universality class. But in contrast to the directed BTW model the height configuration of the directed Zhang model displays no translation invariance [31]. Similar to the center-seeded DBTW model one has to be careful to apply the universality hypothesis of Ben-Hur and Biham.

In summary we reconsidered a deterministic version of the Bak-Tang-Wiesenfeld sandpile model where the sand grains are added always to the central site of the lattice. Similar to the usual BTW model the *Abelian sandpile* formalism allows to calculate some of the static properties of the system. Our numerical investigations show that the deterministic central-seeded model with square symmetric initial conditions exhibits a different scaling behavior than the BTW model, in contrast to the deterministic model without square symmetry.

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References

1. P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
2. P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
3. G. Grinstein, in *Scale Invariance, Interfaces and Non-Equilibrium Dynamics*, edited by A. McKane *et al.*, NATO ASI Series B: Physics Vol. 344 (Plenum Press, 1995).
4. P. Bak, *How Nature Works* (Springer, New York, 1996).
5. M. Schmittmann, R.K.P. Zia, in *Phase Transition and Critical Phenomena*, edited by C. Domb, J.L. Lebowitz, (Academic Press, London, 1995), Vol. 17.
6. D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
7. S.N. Majumdar, D. Dhar, J. Phys. A **24**, L357 (1991), Physica A **185**, 129 (1992).
8. V.B. Priezzhev, J. Stat. Phys. **74**, 955 (1994).
9. E.V. Ivashkevich, J. Phys. A **76**, 3643 (1994).
10. P. Grassberger, S.S. Manna, J. Phys. France **51**, 1077 (1990).
11. S.S. Manna, J. Stat. Phys. **59**, 509 (1990).
12. S. Lübeck, K.D. Usadel, Phys. Rev. E **55**, 4095 (1997).
13. S. Lübeck, K.D. Usadel, Phys. Rev. E **56**, 5138 (1997).
14. A. Chessa, E. Marinari, A. Vespignani, S. Zapperi, Phys. Rev. E **57**, R6241 (1998).
15. S. Lübeck, Phys. Rev. E **58**, 2957 (1998).
16. S.P. Obukhov, in *Random Fluctuations and Pattern Growth*, edited by H.E. Stanley, N. Ostrowsky (Kluwer, Dordrecht, Netherlands, 1988).
17. A. Corral, A. Díaz-Guilera, Phys. Rev. E **55**, 2434 (1997).
18. K. Wiesenfeld, J. Theiler, B. McNamara, Phys. Rev. Lett. **65**, 949 (1990).
19. A. Ben-Hur, O. Biham, Phys. Rev. E **53**, R1317 (1996).
20. M. Markošová, P. Markoš, Phys. Rev. A **46**, 3531 (1992).
21. K. Christensen, Z. Olami, Phys. Rev. E **48**, 3361 (1993).
22. E.V. Ivashkevich, D.V. Ktitarev, V.B. Priezzhev, Physica A **209**, 347 (1994).
23. V.B. Priezzhev, D.V. Ktitarev, E.V. Ivashkevich, Phys. Rev. Lett. **76**, 2093 (1994).
24. J. Krug, Phys. Rev. Lett. **67**, 1882 (1991).
25. S.S. Manna, J. Phys. A **24**, L363 (1991).
26. L.P. Kadanoff, S.R. Nagel, L. Wu, S.M. Zhou, Phys. Rev. A **39**, 6524 (1989).
27. Y.-C. Zhang, Phys. Rev. Lett. **63**, 470 (1989).
28. S. Lübeck, Phys. Rev. E **56**, 1590 (1997).
29. A. Chessa, H.E. Stanley, A. Vespignani, S. Zapperi, Phys. Rev. E **59**, 12 (1999).
30. S. Lübeck, Phys. Rev. E **61**, 204 (2000).
31. S. Lübeck (unpublished).
32. D. Dhar, R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989).